10. Nekonečná geometrická řada

Příklady

① Dána geometrická posloupnost $(a_n)_{n=1}^{\infty}$, $a_n = \left(\frac{1}{2}\right)^n$:

a) Určete prvních 6 členů, kvócient a limitu

$$a_n = \left(\frac{1}{2}\right)^n = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1}$$

GP: $a_n = \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1} \Rightarrow a_1 = \frac{1}{2},\ q = \frac{1}{2}$

$a_n = a_1 q^{n-1}$

$|q|<1 \Rightarrow \lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$

podmínka konvergence

členy: $a_1 = \frac{1}{2}$ $a_2 = \frac{1}{4}$ $a_3 = \frac{1}{8}$ $a_4 = \frac{1}{16}$ $a_5 = \frac{1}{32}$ $a_6 = \frac{1}{64}$

b) vytvořte novou posloupnost $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$ a zjistěte, zda je konvergentní.

$$\left.\begin{aligned}
s_1 &= a_1 = \tfrac{1}{2}\\
s_2 &= a_1 + a_2 = \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4}\\
s_3 &= a_1 + a_2 + a_3 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8}\\
s_4 &= a_1 + a_2 + a_3 + a_4 = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{15}{16}
\end{aligned}\right\}$$

POSLOUPNOST ČÁSTEČNÝCH SOUČTŮ $(s_n)_{n=1}^{\infty}$

$\vdots$

$$s_n = a_1 + a_2 + \ldots + a_n = \underbrace{a_1\frac{q^n-1}{q-1}}_{\text{součet } n \text{ členů GP}} = \frac{1}{2}\,\frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2}-1} = \frac{1}{2}\,\frac{\left(\frac{1}{2}\right)^n - 1}{-\frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \left(\frac{1}{2}\right)^n\right) = \lim_{n\to\infty} 1 - \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 1 - 0 = \underline{\underline{1}}$$

DEFINICE

Nechť je dána posloupnost $(a_n)_{n=1}^{\infty}$

NEKONEČNOU ŘADOU nazýváme výraz $a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum\limits_{n=1}^{\infty} a_n$

ČLENY nekonečné řady jsou členy dané posloupnosti.

VĚTA

NEKONEČNÁ ŘADA je KONVERGENTNÍ $\Leftrightarrow$ posloupnost částečných součtů $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$, JE KONVERGENTNÍ a SOUČET TÉTO ŘADY

$$s = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n$$

DEFINICE

Je-li $(a_n)_{n=1}^{\infty}$ GEOMETRICKÁ POSLOUPNOST s kvócientem q $(a_1 \neq 0)$, pak příslušnou nekonečnou řadu

$$\sum_{n=1}^{\infty} a_n = a_1 + a_1 q + a_1 q^2 + \ldots + a_1 q^{n-1} + \ldots$$

nazýváme NEKONEČNÁ GEOMETRICKÁ ŘADA

VĚTA

je-li $(a_n)_{n=1}^{\infty}$ GEOMETRICKÁ POSLOUPNOST s kvócientem q, pak NEKONEČNÁ GEOMETRICKÁ ŘADA je pro $|q|<1$ $(a_1 \neq 0)$ KONVERGENTNÍ a její součet je $s = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \frac{a_1}{1-q}$

[pro $|q| \geq 1$ je DIVERGENTNÍ]

[důkaz:
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} a_1 \frac{q^n-1}{q-1} = \lim_{n\to\infty} \underbrace{\frac{a_1}{q-1}}_{C\in R} \cdot \lim_{n\to\infty} (q^n-1) =$$
$$= \lim_{n\to\infty} \underbrace{\frac{a_1}{q-1}}_{C\in R} \left(\underbrace{\lim_{n\to\infty} q^n}_{=0 \text{ pro } |q|<1} - \lim_{n\to\infty} 1\right) = \frac{a_1}{q-1}(0-1) = -\frac{a_1}{q-1} = \frac{a_1}{1-q} \text{ cbd.}$$
]

př. $\left(\left(-\frac{1}{3}\right)^{n-1}\right)_{n=1}^{\infty}$ $\quad a_n = \left(-\frac{1}{3}\right)^{n-1} = 1\cdot\left(-\frac{1}{3}\right)^{n-1}$ $\quad a_1 = 1$ $\quad a_3 = \left(-\frac{1}{3}\right)^2 = \frac{1}{9}$

$a_n = a_1 q^{n-1}$ $\quad a_2 = \left(-\frac{1}{3}\right)^{2-1} = -\frac{1}{3}$ $\quad a_4 = \left(-\frac{1}{3}\right)^3 = -\frac{1}{27}$

GP: $a_1 = 1,\ q = -\frac{1}{3}$

$|q| < 1$ PODMÍNKA KONVERGENCE PLATÍ

=> KONVERGUJE I NEKONEČNÁ GEOMETRICKÁ ŘADA a její SOUČET je $s = \lim_{n\to\infty} s_n = \frac{a_1}{1-q}$

$$s = \sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^{n-1} = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots = \frac{1}{1-\left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

př. $(2n)_{n=1}^{\infty}$

$\sum_{n=1}^{\infty} 2n = \underbrace{2+4+6+8+\ldots}_{\text{DIVERGENTNÍ}}$

Příklady

② Dána geom. posloupnost $((-1)^n)_{n=1}^{\infty}$. Urči několik členů posloupnosti $(s_n)_{n=1}^{\infty}$, kde $s_n = a_1 + a_2 + \ldots + a_n$ a dále zjisti, zda je tato posloupnost částečných součtů konvergentní.

GP: $a_1 = -1$ $\quad a_2 = 1$ $\quad a_3 = -1$ $\quad a_4 = 1$ $\qquad q = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots = -1$

podmínka konvergence $|q| < 1$ NEPLATÍ => $((-1)^n)_{n=1}^{\infty}$ DIVERG.

=> $(s_n)_{n=1}^{\infty}$ je také divergentní

$(s_n)_{n=1}^{\infty}$
$$\left.\begin{aligned} s_1 &= a_1 = -1 \\ s_2 &= a_1 + a_2 = -1+1 = 0 \\ s_3 &= a_1+a_2+a_3 = -1+1-1 = -1 \\ s_4 &= a_1+a_2+a_3+a_4 = -1+1-1+1 = 0 \\ &\vdots \end{aligned}\right\} \Rightarrow \left.\begin{aligned} s_{2k-1} &= -1 \\ s_{2k} &= 0 \end{aligned}\right\}$$
posloupnost částečných součtů $(s_n)_{n=1}^{\infty}$ je DIVERGENTNÍ

③ Zjistit, zda je nekonečná řada konvergentní a určit její součet

a) $\sum\limits_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \ldots = 1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots}_{GŘ}$

$\left[1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \quad GP: a_1 = 1 \quad q = \frac{1}{2} \Rightarrow |q| < 1 \text{ konv.} \right]$

GŘ: $a_1 = 1, \quad q = \frac{1}{2}$
$|q| < 1$ podm. konv. platí $\Rightarrow$

$\Rightarrow s = \frac{a_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

b) $\sum\limits_{n=1}^{\infty} 10^{-n} = 10^{-1} + 10^{-2} + 10^{-3} + \ldots = \underbrace{\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3}} + \ldots = \sum\limits_{n=1}^{\infty} \left(\frac{1}{10}\right)^n$

GŘ: $q = \frac{a_2}{a_1} = \frac{\frac{1}{100}}{\frac{1}{10}} = \frac{10}{100} = \frac{1}{10}$

$a_1 = \frac{1}{10}$ $\qquad |q| < 1$ podmínka konvergence GŘ platí $\Rightarrow$

$\Rightarrow s = \frac{a_1}{1-q} = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{10}{90} = \frac{1}{9}$

tedy: $s = 0{,}1 + 0{,}01 + 0{,}001 + \ldots = 0{,}1111\ldots = 0{,}\overline{1}$ $\Big\} \Rightarrow 0{,}\overline{1} = \frac{1}{9}$

c) $\sum\limits_{n=1}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n = \sum\limits_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n = \underbrace{\left(-\frac{2}{3}\right)^1 + \left(-\frac{2}{3}\right)^2 + \left(-\frac{2}{3}\right)^3} + \ldots$

DÚ

GŘ: $q = -\frac{2}{3} \quad a_1 = -\frac{2}{3}$
$|q| < 1$ platí podm. konv.

$\Rightarrow s = \frac{a_1}{1-q} = \frac{-\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = \frac{-\frac{2}{3}}{\frac{5}{3}} = -\frac{6}{15} = -\frac{2}{5}$

d) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{5}}{2^{n-1}} = \frac{\sqrt{5}}{2^0} + \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2^2} + \frac{\sqrt{5}}{2^3} + \ldots = \sqrt{5} + \sqrt{5}\cdot 2^{-1} + \sqrt{5}\cdot 2^{-2} + \sqrt{5}\cdot 2^{-3} + \ldots$

1. zp. GŘ: $q = \frac{\frac{\sqrt{5}}{2}}{\sqrt{5}} = \frac{\frac{\sqrt{5}}{2^2}}{\frac{\sqrt{5}}{2}} = \frac{1}{2}$ podm. konv. $|q| < 1$ platí

$\Rightarrow s = \lim\limits_{n\to\infty} s_n = \frac{a_1}{1-q} = \frac{\sqrt{5}}{1-\frac{1}{2}} = \frac{\sqrt{5}}{\frac{1}{2}} = 2\sqrt{5}$

2. zp. upravíme

$\sum\limits_{n=1}^{\infty} \frac{\sqrt{5}}{2^{n-1}} = \sqrt{5} + \sqrt{5}\cdot 2^{-1} + \sqrt{5}\cdot 2^{-2} + \ldots$

$= \sqrt{5}\left(1 + 2^{-1} + 2^{-2} + \ldots\right) = \sqrt{5}\left(\underbrace{\left(\tfrac{1}{2}\right)^0 + \left(\tfrac{1}{2}\right)^1 + \left(\tfrac{1}{2}\right)^2 + \ldots}\right)$

GŘ: $a_1 = 1 \left[\left(\frac{1}{2}\right)^0\right]$
$q = \frac{1}{2} \quad |q| < 1$ pl.

$\Rightarrow s = \sqrt{5}\, s' = \sqrt{5}\cdot 2 = 2\sqrt{5}$

$s' = \frac{a_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

④ Napište ve tvaru zlomku s celočíselným čitatelem a jmenovatelem

a) $0,\overline{4} = 0,4444\ldots = 0,4 + 0,04 + 0,004 + 0,0004 + \ldots =$

1. způs. $= \underbrace{4\cdot 10^{-1} + 4\cdot 10^{-2} + 4\cdot 10^{-3} + 4\cdot 10^{-4} + \ldots}_{GŘ}$

$q = \frac{4\cdot 10^{-2}}{4\cdot 10^{-1}} = 10^{-1} = \frac{1}{10}$ $s = \frac{a_1}{1-q} = \frac{\frac{4}{10}}{1-\frac{1}{10}} = \frac{\frac{4}{10}}{\frac{9}{10}} = \frac{40}{90} = \frac{4}{9}$

$a_1 = 4\cdot 10^{-1}$ $|q|<1$ platí

$0,\overline{4} = \frac{4}{9}$

lze i s využ. př. ③ b)

2. způs. $0,\overline{4} = 0,4444\ldots = 0,4 + 0,04 + 0,004 + 0,0004 + \ldots =$

$= 4(\underbrace{0,1 + 0,01 + 0,001 + 0,0001 + \ldots}_{GŘ}) =$

$q = \frac{0,01}{0,1} = 0,1 = \frac{1}{10}$

$0,\overline{4} = 4s = 4\cdot\frac{1}{9}$ $a_1 = 0,1$ $s = \frac{a_1}{1-q} = \frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$

$0,\overline{4} = \frac{4}{9}$

b) $-0,3\overline{25} = -(0,3 + 0,025 + 0,00025 + 0,0000025 + \ldots)$

$= -\left(\frac{3}{10} + \frac{25}{1000} + 25\cdot 10^{-5} + 25\cdot 10^{-7} + \ldots\right)$

$= -\left[\frac{3}{10} + 25(\underbrace{10^{-3} + 10^{-5} + 10^{-7} + \ldots}) \right] = -\left(\frac{3}{10} + 25\cdot\frac{1}{990}\right) =$

$\Big[$ GŘ: $q = 10^{-2}$ $|q|<1$ podm. konv. platí

$s = \frac{a_1}{1-q} = \frac{10^{-3}}{1-10^{-2}} = \frac{\frac{1}{1000}}{1-\frac{1}{100}} = \frac{\frac{1}{1000}}{\frac{99}{100}} = \frac{100}{99\cdot 1000} = \frac{1}{990}$ $\Big]$

$= -\frac{3\cdot 99 + 25}{990} = -\frac{297+25}{990} = -\frac{322}{990} = -\frac{161}{495}$

[v 1. roč. čtyř. (5. roč. osmil.)

$0,3\overline{25} = 0,32525\ldots = \frac{3}{10} + 0,02525\ldots$

$a = 0,025|25\ldots$

$1000a = 25,2525\ldots$

$10a = 0,2525\ldots$ $\}\ominus$

$990a = 25$

$a = \frac{25}{990}$]

c) $-2,\overline{5} = -(2 + 0,5 + 0,05 + 0,005 + \ldots) = -2 - (\underbrace{5\cdot 10^{-1} + 5\cdot 10^{-2} + 5\cdot 10^{-3} + \ldots}_{GŘ}) = -2 - s$

$-2,\overline{5} = -2 - \frac{5}{9} = -\frac{23}{9}$

$q = 10^{-1}$, $a_1 = 5\cdot 10^{-1}$

$|q|<1$ pl.

$s = \frac{a_1}{1-q} = \frac{5\cdot 10^{-1}}{1-\frac{1}{10}} = \frac{\frac{5}{10}}{\frac{9}{10}} = \frac{50}{90} = \frac{5}{9}$

d) $-0,8\overline{4} = -(0,8 + 0,04 + 0,004 + 0,0004 + \ldots) = -(0,8 + \underbrace{4\cdot 10^{-2} + 4\cdot 10^{-3} + \ldots}_{GŘ}) = -(0,8 + s)$

DÚ

$q = 10^{-1} = \frac{1}{10}$ $a_1 = 4\cdot 10^{-2} = \frac{4}{100}$

$-0,8\overline{4} = -\left(\frac{8}{10} + \frac{4}{90}\right) = -\frac{72+4}{90} = -\frac{76}{90}$

$|q|<1$ pl.

$-0,8\overline{4} = -\frac{38}{45}$

$s = \frac{a_1}{1-q} = \frac{\frac{4}{100}}{1-\frac{1}{10}} = \frac{\frac{4}{100}}{\frac{9}{10}} = \frac{40}{900} = \frac{4}{90}$

c) $5{,}4\overline{87} = 5 + \frac{4}{10} + 87 \cdot 10^{-3} + 87 \cdot 10^{-5} + \dots =$

ř:

$= \frac{54}{10} + 87(\underbrace{10^{-3} + 10^{-5} + \dots}) = \frac{54}{10} + \frac{87 \cdot 1}{990} = \frac{54 \cdot 99 + 87}{990} = \frac{5433}{990} = \frac{1811}{330}$

$\left[\begin{array}{l} GŘ: \ q = 10^{-2} \quad |q| < 1 \text{ podm. konv. platí} \\ s = \frac{a_1}{1-q} = \frac{10^{-3}}{1-10^{-2}} = \frac{\frac{1}{1000}}{1-\frac{1}{100}} = \frac{\frac{1}{1000}}{\frac{99}{100}} = \frac{100}{99 \cdot 1000} = \frac{1}{990} \end{array} \right]$

nebo

$5{,}4\overline{87} = \frac{54}{10} + \underbrace{87 \cdot 10^{-3} + 87 \cdot 10^{-5} + 87 \cdot 10^{-7} + \dots} = \frac{54}{10} + \frac{29}{33} = \frac{54 \cdot 33 + 290}{330} = \frac{1811}{330}$

$\left[\begin{array}{l} GŘ: \ q = 10^{-2} \quad |q| < 1 \text{ podm. konv. platí} \\ s = \frac{a_1}{1-q} = \frac{87 \cdot 10^{-3}}{1-10^{-2}} = \frac{\frac{87}{1000}}{1-\frac{1}{100}} = \frac{\frac{87}{1000}}{\frac{99}{100}} = \frac{87 \cdot 100}{99 \cdot 1000} = \frac{87}{99} = \frac{29}{33} \end{array} \right]$

⑤ Vypočítejte hodnotu

a) $y = x \cdot \sqrt{x^3} \cdot \sqrt[4]{x^3} \cdot \sqrt[8]{x^3} \cdot \sqrt[16]{x^3} \dots \qquad D = R$

$y = x^1 \cdot x^{\frac{3}{2}} \cdot x^{\frac{3}{4}} \cdot x^{\frac{3}{8}} \cdot x^{\frac{3}{16}} \cdot \dots$

$y = x^{1 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \dots}$

$y = x^{1 + 3(\underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots}_{GŘ})} = x^{1+3s} = x^{1+3\cdot 1} = x^4$

$\left[\begin{array}{l} a_1 = \frac{1}{2} \quad q = \frac{1}{2} \quad |q| < 1 \text{ pl.} \\ s = \frac{a_1}{1-q} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \end{array} \right]$

b) $y = 3\sqrt{3} \cdot \sqrt[4]{3} \cdot \sqrt[8]{3} \cdot \sqrt[16]{3} \cdot \dots$

ř:

$y = 3 \cdot 3^{\frac{1}{2}} \cdot 3^{\frac{1}{4}} \cdot 3^{\frac{1}{8}} \cdot 3^{\frac{1}{16}} \dots = 3^{1 + \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots}_{GŘ}} = 3^{1+s} = 3^2 = 9$

$\left[\begin{array}{l} a_1 = \frac{1}{2} \quad q = \frac{1}{2} \quad |q| < 1 \text{ pl.} \\ s = \frac{a_1}{1-q} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \end{array} \right]$

c) $y = \frac{1+2+3+4+5+\dots+n}{n + \frac{n}{2} + \frac{n}{4} + \dots} = \frac{\frac{n}{2}(1+n)}{2n} = \frac{n(n+1)}{2 \cdot 2n} = \frac{n+1}{4} = \frac{1}{4}(n+1)$

ř:

$\left[\begin{array}{l} 1+2+3+\dots+n \\ \quad AP \\ s_n = \frac{n}{2}(a_1 + a_n) \\ s_n = \frac{n}{2}(1+n) \end{array} \right]$

$\left[\begin{array}{l} n + \frac{n}{2} + \frac{n}{4} + \dots = n(\underbrace{1 + \frac{1}{2} + \frac{1}{4} + \dots}) = \\ = n \cdot 2 = 2n \end{array} \quad \left[\begin{array}{l} GŘ: a_1 = 1, \ q = \frac{1}{2} \\ \quad |q| < 1 \text{ platí} \\ s = \frac{a_1}{1-q} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 \end{array} \right] \right]$

⑥ Zjistěte, pro která $x \in R$ jsou řady konvergentní a určete jejich součet

a) $\sum_{n=1}^{\infty} (1-5x)^n = (1-5x)^1 + (1-5x)^2 + (1-5x)^3 + (1-5x)^4 + \dots$

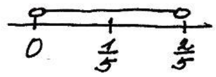
GŘ: $q = \frac{(1-5x)^2}{(1-5x)} = 1-5x$ podm. konvg. $|q| < 1$

$|1-5x| < 1$

$|5x-1| < 1 \quad | : 5$!

$|x - \frac{1}{5}| < \frac{1}{5}$

$x_0 = \frac{1}{5}$



$s = \frac{a_1}{1-q} = \frac{1-5x}{1-(1-5x)} = \frac{1-5x}{5x} \quad \Leftarrow \quad x \in (0, \frac{2}{5})$

b) DÚ $\sum_{n=1}^{\infty} \left(\frac{1}{x}\right)^n = \underbrace{\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}}_{} + \dots = x^{-1} + x^{-2} + x^{-3} + \dots \qquad x \neq 0$

GŘ: $q = \frac{\frac{1}{x^2}}{\frac{1}{x}} = \frac{x}{x^2} = \frac{1}{x}$ podm. konvg. $|q| < 1$

$|\frac{1}{x}| < 1$

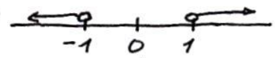
$\frac{1}{|x|} < 1 \quad | \cdot |x|_{>0}$

$s = \frac{a_1}{1-q} = \frac{\frac{1}{x}}{1-\frac{1}{x}} = \left(\frac{\frac{1}{x}}{\frac{x-1}{x}}\right) = \frac{x}{x(x-1)} = \frac{1}{x-1}$

$1 < |x|$

$|x| > 1$

$x_0 = 0$



$x \in (-\infty, -1) \cup (1, +\infty)$

podmínka konv. pro x

⑦ Řešte v R

a) $\sum_{n=1}^{\infty} \left(\frac{2}{x}\right)^{n-1} = \frac{4x-3}{3x-4} \qquad 0 = R, \ D' = R - \{0, \frac{4}{3}\} \qquad [3x-4 \neq 0, \ x = \frac{4}{3}]$

$\underbrace{1 + \frac{2}{x} + \left(\frac{2}{x}\right)^2 + \left(\frac{2}{x}\right)^3 + \dots}_{} = \frac{4x-3}{3x-4} \qquad \left[\left(\frac{2}{x}\right)^{1-1} = \left(\frac{2}{x}\right)^0 = 1 \text{ pro } x \neq 0\right]$

GŘ: $q = \frac{2}{x}$, $a_1 = 1 \rightarrow$ podm. konv. $|q| < 1$ pl.

$s = \frac{a_1}{1-q} = \frac{1}{1-\frac{2}{x}} = \frac{1}{\frac{x-2}{x}}$

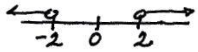
$s = \frac{x}{x-2}$

$|\frac{2}{x}| < 1$

$\frac{2}{|x|} < 1 \quad | \cdot |x|_{>0}$

$2 < |x|$

$|x| > 2$



$x \in (-\infty, -2) \cup (2, +\infty) \cap D'$

$D = (-\infty, -2) \cup (2, +\infty)$

- řa má tvar: $\frac{x}{x-2} = \frac{4x-3}{3x-4}$

$x(3x-4) = (4x-3)(x-2)$

$3x^2 - 4x = 4x^2 - 8x - 3x + 6$

$0 = x^2 - 7x + 6$

$0 = (x-6)(x-1)$

$x_1 = 6 \in D \qquad x_2 = 1 \notin D \qquad K = \{6\}$

b) $\frac{8}{x+10} = 1 \underbrace{- \frac{3}{x} + \frac{9}{x^2} - \frac{27}{x^3} + \dots}$

$0 = \mathbb{R}$
$\mathcal{D}' = \mathbb{R} - \{0, -10\}$ $\quad \left[x \neq 0 \quad x+10 \neq 0 ,\ x \neq -10 \right]$

GŘ: $a_1 = 1,\ q = -\frac{3}{x}$

podm. konv: $|q| < 1$

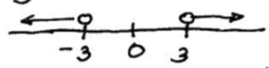
$\left|-\frac{3}{x}\right| < 1$

$\frac{3}{|x|} < 1$

$3 < |x|$

$|x| > 3$

$[x_0 = 0]$



$\mathcal{D}'' = (-\infty, -3) \cup (3, +\infty)$

$s = \frac{a_1}{1-q}$

$s = \frac{1}{1-\left(-\frac{3}{x}\right)} = \frac{1}{1+\frac{3}{x}} = \frac{1}{\frac{x+3}{x}} = \frac{x}{x+3}$

$\mathcal{D} = \mathcal{D}' \cap \mathcal{D}'' = (-\infty, -10) \cup (-10, -3) \cup (3, +\infty)$

– rce má tvar

$\frac{8}{x+10} = \frac{x}{x+3}$

$8(x+3) = x(x+10)$

$8x + 24 = x^2 + 10x$

$x^2 + 2x - 24 = 0$

$(x+6)(x-4) = 0$

$x_1 = -6 \in \mathcal{D}$ $\quad x_2 = 4 \in \mathcal{D}$

$\mathcal{K} = \{-6, 4\}$

c) $\sum_{n=1}^{\infty} (3x)^{n-1} = 10$ $\qquad \mathcal{D}' = \mathbb{R}$

$(3x)^0 + (3x)^1 + (3x)^2 + \dots = 10$

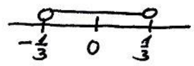
$1 + \underbrace{3x + 9x^2 + \dots} = 10$

GŘ: $q = \frac{9x^2}{3x} = \frac{3x}{1} = 3x$ $\qquad$ podm. konv: $|q| < 1$

$|3x| < 1$

$3|x| < 1$

$|x| < \frac{1}{3}$



$s = \frac{a_1}{1-q} = \frac{1}{1-3x}$ $\qquad \Leftarrow \qquad x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

$\mathcal{D}'' = \left(-\frac{1}{3}, \frac{1}{3}\right)$

rovnice má tvar

$\frac{1}{1-3x} = 10$ $\qquad \mathcal{D} = \mathcal{D}' \cap \mathcal{D}'' = \left(-\frac{1}{3}, \frac{1}{3}\right)$

$1 = 10(1-3x)$

$1 = 10 - 30x$

$30x = 9$

$x = \frac{9}{30} = \frac{3}{10} \in \mathcal{D}$ $\qquad \mathcal{K} = \left\{\frac{3}{10}\right\}$

d) $\underbrace{1 - x + x^2 - x^3 + \dots} = \frac{\sqrt{2}}{2}$

$0 = \mathbb{R}$
$\mathcal{D}' = \mathbb{R}$

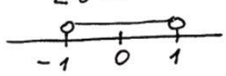
GŘ: $a_1 = 1,\ q = \frac{-x}{1} = \frac{x^2}{-x} = -x$

podm. konv: $|q| < 1$

$|-x| < 1$

$|x| < 1$

$[x_0 = 0]$



$\mathcal{D}'' = (-1, 1)$

$\mathcal{D} = \mathcal{D}' \cap \mathcal{D}'' = \mathbb{R} \cap (-1, 1) = (-1, 1)$

$s = \frac{a_1}{1-q} = \frac{1}{1-(-x)} = \frac{1}{1+x}$

– rce má tvar $\qquad \mathcal{D} = (-1, 1)$

$\frac{1}{1+x} = \frac{\sqrt{2}}{2}$

$2 = \sqrt{2}(1+x)$

$2 = \sqrt{2} + \sqrt{2} \cdot x$

$\sqrt{2}\, x = 2 - \sqrt{2}$

$x = \frac{2-\sqrt{2}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}(2-\sqrt{2})}{2}$

usměrnit

$x = \frac{2\sqrt{2} - 2}{2} = \frac{2(\sqrt{2}-1)}{2}$

$x = \sqrt{2} - 1 \in \mathcal{D}$ $\quad [\sqrt{2} - 1 \doteq 1{,}4 - 1 = 0{,}4 \in \mathcal{D}]$

$\mathcal{K} = \{\sqrt{2} - 1\}$

e) $x+3x^2+x^3+3x^4+\dots=\frac{5}{3}$

ř:

rozložíme na 2 řady

$x+x^3+x^5+\dots$

GŘ: $a_1=x \quad q=x^2 \quad |x^2|<1$

$|x|<1$

$s_1=\frac{x}{1-x^2} \quad x\in(-1,1) \quad D''\ (-1,1)$

$3x^2+3x^4+3x^6+\dots$

$3(\underbrace{x^2+x^4+x^6+\dots}_{})$

GŘ: $a_1=x^2 \quad q=x^2 \quad |x^2|<1$

$|x|<1$

$s_2=3\frac{x^2}{1-x^2} \quad x\in(-1,1)$

$D'''\ (-1,1)$

$D'=R \qquad D=D'\cap D''\cap D'''$

$D=(-1,1)$

rce má tvar: $s_1+s_2=\frac{5}{3}$

$$\frac{x}{1-x^2}+\frac{3x^2}{1-x^2}=\frac{5}{3}$$

$$3x+9x^2=5-5x^2$$

$$14x^2+3x-5=0$$

$$x_{1,2}=\frac{-3\pm\sqrt{9+4\cdot14\cdot5}}{28}=\frac{-3\pm\sqrt{289}}{28}$$

$$=\frac{-3\pm17}{28}= \begin{cases} -\frac{20}{28}=-\frac{5}{7}\in D \\ \frac{14}{28}=\frac{1}{2}\in D \end{cases}$$

$K=\left\{\frac{1}{2},-\frac{5}{7}\right\}$

8 Řešte v R

a) $\log x+\log\sqrt{x}+\log\sqrt[4]{x}+\log\sqrt[8]{x}+\dots=2 \qquad D'=R^+=(0,+\infty) \quad [\underset{\text{pro }\log x}{x>0}\wedge\underset{\text{pro }\sqrt{}}{x\geq0}]$

$\log x^1+\log x^{\frac12}+\log x^{\frac14}+\log x^{\frac18}+\dots=2 \quad [\log_z x^n=n\log_z x] \qquad D=D'\cap D''=(0,+\infty)$

1. zp.

$1\log x+\frac12\log x+\frac14\log x+\frac18\log x+\dots=2$ – rce má tvar

$\log x(\underbrace{1+\frac12+\frac14+\frac18+\dots}_{})=2$

GŘ: $a_1=1,\quad q=\frac12 \quad \left(=\frac{\frac14}{\frac12}=\frac24=\frac12\right)$

podm. konv. $|q|<1$ platí pro $\forall x\in R \Rightarrow D''=R$

$s=\frac{a_1}{1-q}=\frac{1}{1-\frac12}=\frac{1}{\frac12}=2$

$s\cdot\log x=2 \qquad [\log_z x=y\iff z^y=x]$

$2\log x=2$

$\log x=1 \qquad [\log_{10}x=1\iff10^1=x]$

$x=10\in D$

$K=\{10\}$

lze i 2. zp.

$\log x+\frac12\log x+\frac14\log x+\dots=2$

GŘ: $a_1=\log x \qquad q=\frac{\frac12\log x}{\log x}=\frac12$

podm. konv. $|q|<1$ platí pro $\forall x\in R^+$

$\rightarrow s=\frac{a_1}{1-q}=\frac{\log x}{1-\frac12}=\frac{\log x}{\frac12}$

$s=2\log x$

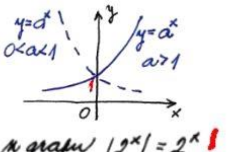
rce má tvar $2\log x=2\dots$ viz výše

b) $2^x+4^x+8^x+16^x+\dots=1 \qquad D'=R$

$2^x+(2^2)^x+(2^3)^x+(2^4)^x+\dots=1$

$\underbrace{2^x+2^{2x}+2^{3x}}_{}+2^{4x}+\dots=1$

GŘ: $a_1=2^x \qquad q=\frac{2^{2x}}{2^x}=\frac{2^x\cdot2^x}{2^x}=2^x$



podm. konv. $|q|<1$

$|2^x|<1$ z grafu $|2^x|=\underset{>0}{2^x}$!

$2^x<1$

$2^x<2^0$

$x<0$ upravujeme – pro základ $a=2>1$! NEMĚNÍME ZNAK NEROVNOSTI

$D''=R^-=(-\infty,0)$

$D=D'\cap D''=(-\infty,0)$

$s=\frac{a_1}{1-q}=\frac{2^x}{1-2^x}$

$D=(-\infty,0)$

-rce má tvar

$$\frac{2^x}{1-2^x}=1$$

$$2^x=1-2^x$$

$$2^x+2^x=1$$

$$2\cdot2^x=1$$

$$2^{x+1}=1$$

$$2^{x+1}=2^0$$

$$x+1=0$$

$$x=-1\in D$$

$K=\{-1\}$